THE TEMPERATURE FIELD OF A THIN PLATE UNDER CONDITIONS OF MONOTONIC HEATING

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An approximate solution is given for the heat-conduction equation for the temperature field under conditions of monotonic heating. The solution is obtained by linearization according to the small parameter method.

An examination was made in [1] of the special features of temperature regulation in a monotonically heated (or cooled) plate with a symmetrical temperature field. The analysis used a linear approximation to the thermophysical parameters and the heating rate as a function of temperature within the temperature field of the plate. A generalization of the relations found in [1] for the case of an unsymmetrical temperature field of the plate may be found by similar means.

The problem is formulated as follows. An infinite plate is heated monotonically by a uniform heat flux normal to its edges. There are no internal heat sources. The thermophysical parameters λ and a are monotonic functions of temperature, and the heating rate b (x, τ) is nearly constant. The temperature drop inside the plate does not exceed several tens of degrees, and the temperature field obeys an equation of the form

$$\frac{d^2t}{dx^2}+k_\lambda\left(\frac{dt}{dx}\right)^2=\frac{b}{a},\quad k_\lambda=\frac{1}{\lambda}\,\frac{d\lambda}{dt}\,. \tag{1}$$

Equation (1) is nonlinear. However, in solving it in accordance with the objective described, it is important to establish limits within which the temperature field of the plate obeys, with satisfactory accuracy, the relations of the regular regime of the second kind. In other words, it is important to find the conditions for which the solution of (1) agrees, to an assigned accuracy, with the solution of the equation

$$\frac{d^2t}{dx^2}=\frac{b}{a}\,, \tag{2}$$

in which the thermophysical parameters and the heating rate are assumed to be constant.

The solution of (2) is generally known to be

$$t(x,\ \tau)=t(0,\ \tau)+\vartheta_{l_1-l}\,\frac{x}{2l}+\frac{b}{2a}\,x^2. \tag{3}$$

Taking the foregoing into account, an approximate solution of (1) is feasible, using linearization according to the "small parameter" method [2].

As the small parameter we shall choose the drop in temperature $\vartheta(x,\ \tau)$ in the plate, and we shall represent $\lambda(t)$, $a(t)$ and $b(x,\ \tau)$ as functions of the form

$$\lambda=\lambda_0(1+k_\lambda\vartheta),\ a=a_0(1+k_a\vartheta),\ b=b_0(1+k_b\vartheta), \tag{4}$$

where the coefficients k_λ, k_a and k_b refer to t_0, and at any instant of time τ are considered to be independent of x and $\vartheta(x,\ \tau)$.

Substitution of (4) into (1) allows us to write the latter in the form

$$\frac{d^2t}{dx^2}+k_\lambda\left(\frac{d\vartheta}{dx}\right)^2=\frac{b_0}{a_0}\,\frac{1+k_b\vartheta}{1+k_a\vartheta}\,. \tag{5}$$

Equation (5) may be transposed to a form more convenient for mathematical analysis by introducing the limits

$$k_\lambda\vartheta\leqslant 0.1,\ k_a\vartheta\leqslant 0.1,\ k_b\vartheta\leqslant 0.1. \tag{6}$$

The limits are justified on two considerations. Firstly, if the parameters λ, a, and b may be represented in the neighborhood of t_0 by absolutely convergent power series in ϑ, then the smaller the complexes $k_\lambda\vartheta$, $k_a\vartheta$ and $k_b\vartheta$ are, the more valid are the relations in (4). Secondly, the greatest interest in practice, especially in making thermophysical measurements, attaches precisely to heating with small internal temperature drops.

Taking the inequalities (6) into account, Eq. (5) may be replaced, with an error of not more than 1%, by another equation:

$$\frac{d^2\vartheta}{dx^2}=\frac{b_0}{a_0}\left[1+(k_b-k_a)\vartheta-k_\lambda\,\frac{a_0}{b_0}\left(\frac{d\vartheta}{dx}\right)^2\right]. \tag{7}$$

The complex $k_\lambda(a_0/b_0)(d\vartheta/dx)^2$ appearing in (7), as will be shown below, is commensurate with the complex $k_\lambda\vartheta$, and therefore, in conformity with (6), we have

$$\Delta\sigma=(k_b-k_a)\vartheta-k_\lambda\,\frac{a_0}{b_0}\left(\frac{d\vartheta}{dx}\right)^2\leqslant 0.2. \tag{8}$$

The inequality (8) in turn enables us to state that the parameter $\Delta\sigma$ in (7) is a correction, and introduces a negligible disturbing influence in comparison with an equation of the type of (2)

$$\frac{d^2\vartheta^0}{dx^2}=\frac{b_0}{a_0}\,. \tag{9}$$

Because of this we may further simplify and linearize (7) by finding an approximate correction $\Delta\sigma$ from the solution of (9)

$$\vartheta^0=\vartheta_{l_1-l}\,\frac{x}{2l}+\frac{b_0}{2a_0}\,x^2, \tag{10}$$

which may be considered as a solution of (7) in the first approximation.

From the solution of (10) we have

$$\left(\frac{d\vartheta^0}{dx}\right)^2 = \left(\frac{\vartheta_{l_1-l}}{2l}\right)^2 + \vartheta_{l_1-l}\frac{b_0}{a_0}\frac{x}{l} + \frac{b_0^2}{a_0^2}x^2. \quad (11)$$

After substituting the expressions for ϑ^0 and $(d\vartheta^0/dx)^2$ in (7), we arrive at the equation

$$\frac{d^2\vartheta}{dx^2} = \frac{b_0}{a_0} + (k_b - k_a)\left(\vartheta_{l_1-l}\frac{x}{2l} + \frac{b_0}{2a_0}x^2\right)\frac{b_0}{a_0} - k_\lambda\left[\left(\frac{\vartheta_{l_1-l}}{2l}\right)^2 + \vartheta_{l_1-l}\frac{b_0}{a_0}\frac{x}{l} + \frac{b_0^2}{a_0^2}x^2\right], \quad (12)$$

which is a sufficiently good approximation to (7), and has an exact solution.

Integration of (12) leads to the desired function for the temperature field of the thin plate:

$$\vartheta = \frac{\vartheta_{l_1-l}}{2l}x + \frac{b_0}{2a_0}x^2 - \frac{1}{2}k_\lambda\cdot\left(\frac{\vartheta_{l_1-l}}{2l}\right)^2 x^2 + \frac{1}{3}(2k_\lambda + k_a - k_b)\frac{b_0}{2a_0}\left[\frac{\vartheta_{l_1-l}}{2l}(l^2 - x^2)x - \frac{b_0}{2a_0}\frac{x^4}{2}\right]. \quad (13)$$

Function ϑ from (13) differs from function ϑ^0 found in the first approximation from (11) by the correction

$$\Delta\vartheta = \frac{1}{3}(2k_\lambda + k_a - k_b)\frac{b_0}{2a_0}\times\left[\frac{\vartheta_{l_1-l}}{2l}(l^2 - x^2)x - \frac{b_0}{2a_0}\frac{x^4}{2}\right] - \frac{1}{2}k_\lambda\left(\frac{\vartheta_{l_1-l}}{2l}\right)^2 x^2 \quad (14)$$

and may be further abbreviated to the form

$$\vartheta(x, \tau) = \vartheta^0(x, \tau) + \Delta\vartheta(x, \tau). \quad (15)$$

The correction $\Delta\vartheta$ determines the disturbing influence of the parameters k_λ, k_a and k_b on the nature of the temperature field in the thin plate, and may be used to evaluate the conditions of temperature regulation under conditions of monotonic heating (or cooling). In fact, if the inequality

$$\Delta\vartheta \leqslant 0.01\vartheta^0 \quad (16)$$

is satisfied, then the temperature field in the plate obeys the relations of the regular regime of the second kind [3, 4] with an error not exceeding 1%. The inequality (16) may otherwise be used to establish the limits of applicability of the relations of the regular regime of the second kind under monotonic heating conditions. As the relative value of $\Delta\vartheta$ in function (15) increases, the accuracy of the solution naturally falls. The temperature field of the plate may be calculated with the aid of function (13), if the correction $\Delta\vartheta$ does not exceed $(0.1-0.2)\vartheta^0$.

Function (13) describes a steady quasi-regular heating (or cooling) regime for the plate. To find the duration of the initial, irregular phase of the test we may, in the first approximation, recommend the use of experimental results relating to heating a plate by a constant heat flux [4].

In practice we usually require to know function (13) in conducting thermophysical investigations under monotonic heating conditions, especially in studying the temperature dependence $a(t)$ of materials. The equation for calculating $a(t)$ may be found from function (13), if the temperature drops ϑ_l and ϑ_{-l} are measured directly in the experiment. In fact, a combination of values of ϑ_l and ϑ_{-l} found from (13) leads to quite a convenient equation for practical purposes:

$$a = \frac{b_0 l^2}{\vartheta_l + \vartheta_{-l}}(1 - \Delta\sigma_a), \quad (17)$$

where

$$\Delta\sigma_a = \frac{1}{8}k_\lambda\frac{(\vartheta_l - \vartheta_{-l})^2}{\vartheta_l + \vartheta_{-l}} + \frac{1}{12}(2k_\lambda + k_a - k_b)(\vartheta_l + \vartheta_{-l}). \quad (18)$$

According to the limits assumed above, function $\Delta\sigma_a$ must play the part of a correction in (17). For practical purposes its value in (18) may be found from the approximate relation

$$\vartheta_l + \vartheta_{-l} \approx b_0 l^2/a_0. \quad (19)$$

In experiments to determine $a(t)$ in order to calculate the correction $\Delta\sigma_a$ in addition to experimentally determined values of ϑ_l, ϑ_{-l}, and k_b, we require to know tentative values of the parameters k_λ and k_a. The parameter k_a, in particular, may be found from an approximate calculation of $a(t)$ with the aid of (19). To calculate k_λ, independent measurements are unfortunately necessary.

Calculation of the thermal diffusivity is appreciably simplified, naturally, if the optimum test conditions are chosen for which the correction $\Delta\sigma_a$ turns out to be negligibly small. It is clear from the composition of the correction $\Delta\sigma_a$, that to secure optimum test conditions, we should achieve as symmetrical heating conditions for the plate as possible, with quite small values of ϑ_l and ϑ_{-l}. In this way, expression (18) may be used as an initial condition in the choice of the optimum construction of the calorimeter equipment [5] intended for measuring $a(t)$.

As a second example of application of relation (13) found above for $\vartheta(x, \tau)$, we take the experimental measurement of $\lambda(t)$ of materials under conditions of monotonic heating of a thin plate from one side [5].

The methods of this series amount in themselves to measuring the heat flux and true temperature gradient in the so-called "basic" layer of the plate, in which the true temperature gradient coincides with the mean gradient measured from the temperature drop at the outside edges of the plate. In the special case when the temperature field of the plate obeys relation (10) with sufficient accuracy, the basic layer coincides with the central layer (x = 0). In the general case, however, there is a displacement of the basic layer from the center to some value x_0. Differentiation of function (13) leads to the equation

$$\frac{1}{3}(2k_\lambda + k_a - k_b)\frac{b_0}{2a_0}\left(\frac{b_0}{2a_0}2x_0^3 + \frac{\vartheta_{l_1-l}}{2l}3x_0^2\right) - \left[\frac{b_0}{a_0} - k_\lambda\left(\frac{\vartheta_{l_1-l}}{2l}\right)^2\right]x_0 - \frac{1}{3}(2k_\lambda + k_a - k_b)\frac{b_0}{2a_0}\frac{\vartheta_{l_1-l}}{2l}l^2 = 0. \tag{20}$$

Analysis shows that in the majority of practical problems Eq. (20) admits of a simplified solution, since its first term, containing x_0^3 and x_0^2, is most often very small compared to the other two terms. This conclusion is based on the simplifying inequalities (6) and (16) chosen in solving the problem, an account of which the displacement x_0 does not usually exceed the value

$$x_0 \leqslant 0.1l. \tag{21}$$

The approximate equation giving the displacement of the basic layer, if (21) is used, takes the form

$$x_0 = -\frac{(1/6)(2k_\lambda + k_a - k_b)(\vartheta_l + \vartheta_{-l})(\vartheta_{l_1-l}/2l)}{(\vartheta_l + \vartheta_{-l})/l^2 - k_\lambda(\vartheta_{l_1-l}/2l)^2}. \tag{22}$$

In the more special case, when the inequality

$$k_\lambda\left(\frac{\vartheta_{l_1-l}}{2l}\right)^2 \ll \frac{\vartheta_l + \vartheta_{-l}}{l^2} \tag{23}$$

is satisfied, the expression for x_0 is additionally simplified:

$$x_0 = -(1/12)(2k_\lambda + k_a - k_b)\, l\, \vartheta_{l_1-l}. \tag{24}$$

The temperature of the basic layer, to which the measured thermal conductivity in the problem examined must relate, can be evaluated approximately by the expression derived from (13)

$$t(x_0, \tau) = t(0, \tau) \quad \frac{\vartheta_{l_1-l}}{2l}x_0 + \frac{b_0}{2a_0}x_0^2. \tag{25}$$

NOTATION

$t = t(x, \tau)$) temperature of plate; $2l$) thickness of plate; τ) time; x) variable coordinate reckoned from the central layer; $b = dt/d\tau$, $\lambda = \lambda(t)$, $a = a(t)$) thermal conductivity and diffusivity of plate at t; $\vartheta = t(x, \tau) - t(0, \tau)$; $\vartheta_l = \vartheta(l_1 \tau)$; $\vartheta_{-l} = \vartheta(-l_1 \tau)$; $\vartheta_{l_1-l} = \vartheta_l - \vartheta_{-l}$; $\lambda_0 = \lambda(t_0)$, $a_0 = a(t_0)$ and $b_0 = b(t_0)$ where $t_0 = t(0, \tau)$; $k_\lambda = \frac{1}{\lambda_0}\frac{d\lambda_0}{dt}$, $k_a = \frac{1}{a_0}\frac{da_0}{dt}$, $k_b = \frac{1}{b_0}\frac{db_0}{dt}$) relative temperature coefficients of λ, a, and b at t_0.

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